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Predicting Air Pollutant Spread from Fixed Sources Using Mathematical Techniques

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Abstract:

This paper presents a mathematical framework for analyzing the spread of atmospheric pollutants from fixed sources using the advection-diffusion equation. Utilizing the Gaussian plume model, an analytical solution is derived for steady-state pollutant dispersion under the influence of mean wind velocity and turbulent diffusion. Numerical simulations illustrate how pollutant concentration evolves with increasing downwind distance from the source, exhibiting broader spatial distribution and reduced peak concentration due to atmospheric mixing. At shorter distances (e.g., 500 m), the plume is narrow and highly concentrated near the release height, while at greater distances (e.g., 2000 m), the pollutant becomes more diluted and widely spread in both vertical and lateral directions. These results align with physical expectations and validate the reliability of the model under idealized conditions. The approach provides a foundational methodology for environmental risk assessment, industrial site planning, and air quality regulation. Future extensions could incorporate time-dependent effects, variable meteorological profiles, and reactive pollutant dynamics to improve applicability to real-world atmospheric systems.

Keywords: Pollutant Dispersion, Advection-Diffusion Equation, Gaussian Plume Model, Atmospheric Transport, Steady-State Modeling, Point Source Emission, Wind-Driven Advection, Turbulent Diffusion, Concentration Profile, Analytical Solution, Plume Evolution, Air Pollution Modeling, Environmental Modelling, Atmospheric Pollution, Numerical Simulation.

1. Introduction

Atmospheric pollution has emerged as a pressing and multifaceted global challenge, largely fueled by the accelerated growth of industrialization and the relentless expansion of urban centers [3,12]. As cities grow and industrial activities intensify, the release of harmful pollutants into the atmosphere has reached alarming levels. A significant portion of these pollutants originates from stationary sources such as industrial smokestacks, chimneys, manufacturing plants, power generation facilities, and chemical processing units. These sources emit a continuous stream of hazardous substances, including particulate matter, nitrogen oxides, sulfur dioxide, carbon monoxide, and volatile organic compounds [8,13]. These pollutants not only degrade air quality but also pose severe threats to ecological balance and human health, contributing to respiratory illnesses, cardiovascular diseases, acid rain, and climate change. To effectively mitigate and manage the impact of such emissions, it is imperative to understand how pollutants disperse in the atmosphere under varying meteorological and geographical conditions. This understanding forms the basis for devising informed strategies in environmental protection, urban planning, industrial zoning, and public health policymaking [4,10]. Mathematical modeling plays a crucial role in this endeavor by offering a

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structured and quantitative framework to simulate the dispersion behavior of atmospheric pollutants. It allows researchers and policymakers to predict how pollutants travel through the air, how their concentration varies with time and distance from the source, and how external factors such as wind speed, atmospheric stability, and terrain influence their spread. One of the most fundamental and widely used approaches in this domain is the advection-diffusion equation, which serves as the theoretical foundation for modeling pollutant transport processes [6,9]. This partial differential equation accounts for both the movement of pollutants due to air currents (advection) and their spreading due to turbulence and molecular diffusion (diffusion). A well-known analytical solution to the advection-diffusion equation is the Gaussian plume model, which provides a simplified yet powerful method to estimate pollutant concentrations downwind from a source [5,15]. It assumes steady state conditions and homogenous atmospheric properties, making it particularly useful for regulatory applications. Owing to its mathematical tractability and practical reliability, the Gaussian plume model has been extensively implemented in environmental modeling tools and regulatory frameworks, including the AERMOD (American Meteorological Society/Environmental Protection Agency Regulatory Model) and ISC3 (Industrial Source Complex Model) [7,16]. These models are instrumental in air quality management, helping agencies evaluate compliance with environmental standards, assess the environmental impact of proposed industrial projects, and design pollution control strategies. The integration of mathematical modeling with atmospheric science offers a vital pathway to understanding and combating the far-reaching consequences of air pollution, ensuring the protection of both human life and the environment.

2. Mathematical Formulation of the Model

The diffusion equation for describing the dispersion of atmospheric pollutants in the atmosphere is given as:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}\left(k_x\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right) - \alpha C \tag{1}$$

where x, y, z are spatial coordinates (downwind, crosswind, and vertical), C represents the concentration pollutants in the atmosphere, u, v, and w are the wind speed and k_x, k_y and k_z are diffusivities in x, y, and z direction, α represents the removal rate of atmospheric pollutants naturally.

3. Physical Assumptions

In this study, we analyze pollutant dispersion under steady-state conditions, meaning the pollutant concentration remains constant over time (∂C/∂t = 0). The mean wind is assumed to be constant and unidirectional, blowing solely along the x-axis with a velocity u=U, while the velocity components in the y and z directions are zero (v=w=0). Eddy diffusivity in the x-direction (K_x) is considered negligible ($K_x \approx 0$), whereas diffusivities in the y and z directions (K_y) are assumed to be constant. No chemical reactions or removal mechanisms, such as deposition or transformation, are included in the model. Pollutant emission is assumed to originate from a point source located at a height H above the ground, minimizing the influence of surface-level obstructions [2,17]. The steady, uniform wind facilitates consistent transport of the emitted substances. Gravitational settling and chemical interactions are considered insignificant, allowing for a simplified, non-reactive model. Furthermore, the atmospheric environment is treated as homogeneous, with uniform properties such as temperature, pressure, and density throughout the domain. This assumption supports a more tractable and idealized analysis of pollutant dispersion. A continuous point source at height H above the ground located at x = 0, y = 0, z = H. The emission source is a point source, continuously releasing pollutants at a rate Q from a specific height H above the ground. The terrain is flat, and the ground surface at z=0 reflects the pollutant, implying no loss at the boundary. Furthermore, the concentration of the pollutant is sufficiently dilute such that it does not influence the flow field, thus allowing it to be treated as a passive scalar.

The transport of pollutant concentration C in a steady-state atmosphere is governed by the advection-diffusion equation as below:

$$u(x)\frac{\partial C}{\partial x} = k_y\frac{\partial^2 C}{\partial y^2} + k_z\frac{\partial^2 C}{\partial z^2} - \lambda C + S(x,y,z) \quad (2)$$

where:

- C(x,y,z): Pollutant concentration (kg/m³)
- u(x): Mean wind velocity in the x-direction (m/s)
- K_y, K_z: Eddy diffusion coefficients in y and z directions (m²/s)
- λ: Natural removal or decay rate of pollutant
- S(x,y,z): Source term

4. Boundary and Initial Conditions

The boundary and initial conditions are as below:

$$C \rightarrow 0 \text{ as } \; y, z \rightarrow \infty. \text{ (the pollutant disperses far from the source),} \quad (3)$$

At ground level (z = 0): zero flux condition:

$$\text{No flux at the ground surface: } \frac{\partial C}{\partial z} = 0 \; at \; z = 0 \quad (4)$$

Source term represented as a Dirac delta function at $(x = 0,\; y = 0,\; z = \mathrm{H})$:

$$\mathrm{S(x,y,z)} = \frac{Q\delta(y)\delta(z-\mathrm{H})}{u}, \quad (5)$$

where Q is the emission rate and H is the release height, and δ is the Dirac delta-function.

5. Analytical Solution

Using the method of separation of variables and Green's function, the solution to the steady-state equation is given by:

$$C = \frac{Q}{2\pi\sigma_y\sigma_z u}exp(-\frac{y^2}{2\sigma_y^2})[exp(-\frac{(z-H)^2}{2\sigma_z^2}) + exp(-(\frac{(z+H)^2}{2\sigma_z^2}))] \quad (6)$$

where Q is the pollutant emission rate (kg/s), and σ_y, σ_z are empirical dispersion coefficients, dependent on downwind distance x, atmospheric stability, and turbulence intensity [11,16]. Empirical dispersion coefficients depending on x, approximated using standard empirical formulas:

$$\sigma_y = a(x)^b, \quad \sigma_z = c(x)^d \quad (7)$$

where a=0.22, b=0.89, and c=0.20, d=0.89 are site-specific constants assuming neutral atmospheric conditions.

The Gaussian plume model effectively captures the spatial evolution of pollutant concentration:

- The highest concentration is observed close to the source.
- As distance increases, the peak concentration drops, and the plume becomes more diffused.

- The vertical reflection term $exp(-(\frac{(z+H)^2}{2\sigma_z^2})$ enforces the ground condition.
- The assumption of isotropic turbulence in the lateral direction justifies the symmetric dispersion.

6. Numerical Results and Discussion

This study develops a mathematical framework to analyze the steady-state transport and dispersion of atmospheric pollutants emitted from a continuous point source. The methodology is based on solving the advection-diffusion equation under idealized atmospheric conditions [1,11]. Numerical simulations were conducted for different atmospheric conditions using typical parameter values for Q, u, Ky, Kz. Results indicate:

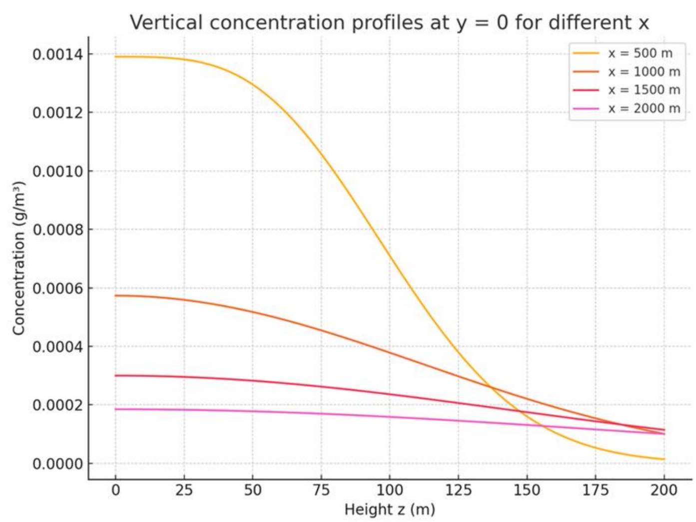

Figure (1): Vertical concentration profile at y=0 for different x

Figure (1) presents the concentration profiles of pollutants at downwind distances of 500 m, 1000 m, 1500 m, and 2000 m. These plots illustrate how the pollutant disperses in both the lateral (y-direction) and vertical (z-direction) directions. The highest concentrations are consistently centered around the emission height of 50 m, with the plume exhibiting a characteristic bell-shaped curve [4,14]. As the distance from the source increases, the plume becomes progressively broader and the peak concentration diminishes. At 500 m, the concentration is tightly focused with a sharp peak near the source height. By 2000 m, the plume spreads extensively, and the pollutant becomes significantly diluted, indicating the influence of atmospheric turbulence on dispersion. The analytical solution derived from the Gaussian plume model was used to simulate the steady-state dispersion of pollutants from a continuous point source. Numerical simulations were carried out at four key downwind distances—500 m, 1000 m, 1500 m, and 2000 m—assuming a constant wind velocity of 5 m/s and a source height of 50 m. Both contour plots and line profiles were generated to visualize the pollutant distribution in the atmosphere. The results provided clear insights into how the plume behaves as it travels away from the source under neutral atmospheric conditions. In terms of vertical and lateral dispersion, the contour plots show that the plume is narrow and highly concentrated around the release height at closer distances (e.g., x = 500 m). As the plume progresses downstream, the pollutant spreads more widely in both directions. This broadening reflects the effect of turbulent diffusion, which promotes mixing and contributes to a symmetrical distribution pattern in the atmosphere. The increasing spread is accompanied by a corresponding decrease in peak concentration, highlighting the dilution process.

Vertical concentration profiles at the centerline (y = 0), as illustrated in Figure 1, further demonstrate the transformation of the plume. At 500 m, the profile is sharply peaked at the emission height, suggesting minimal vertical dispersion. As the downwind distance increases from 1000 m to 2000 m, the profiles become broader and flatter, indicating enhanced mixing and a substantial reduction in concentration. The symmetry observed in these profiles is a consequence of the reflective boundary condition imposed at ground level in the Gaussian plume formulation. The simulations reveal two dominant trends as the downwind distance increases. First, the peak concentration of pollutants consistently decreases, which aligns with the expected dilution effects caused by atmospheric turbulence. Second, the plume becomes more spatially extensive, with increased dispersion in both lateral and vertical directions. These behaviors are consistent with the classical theory of passive tracer dispersion and validate the model's predictive capability under simplified, idealized conditions. Despite its assumptions—such as steady-state conditions, constant wind speed, and the absence of chemical reactions—the model successfully captures the essential dynamics of pollutant dispersion. It accurately predicts spatial concentration distributions, provides valuable insights into the effects of atmospheric mixing, and serves as a reliable first-order estimation tool for air quality assessments. This makes the model particularly useful in the context of preliminary environmental evaluations, industrial site planning, and emergency preparedness for accidental pollutant releases.

7. Conclusion

The presented model effectively captures the fundamental dynamics of pollutant dispersion from a point source in a steady atmospheric environment. Using the Gaussian plume framework, an analytical solution was derived that elucidates how pollutant concentration varies with downwind distance, lateral displacement, and height. Numerical simulations based on this solution demonstrate that:

- **Closer to the source** (e.g., at 500 m), the concentration profile is sharply peaked and narrowly focused on the emission height (50 m).
- **As the distance increases** (up to 2000 m), the plume becomes significantly broader in both the vertical and lateral directions, indicating enhanced mixing due to atmospheric turbulence.
- **Peak concentrations decrease** with distance, confirming the dilution effect as pollutants disperse over a larger volume of air.

The results confirm classical Gaussian dispersion behavior and exhibit physical realism in terms of symmetry, decay patterns, and plume evolution. These findings validate the reliability of the model under idealized conditions and affirm its usefulness for preliminary air quality assessments. Although the model simplifies several real-world factors—such as atmospheric stratification, chemical reactions, topographical complexity, and temporal variability—it provides a robust and analytically tractable foundation for more advanced studies. Future extensions could integrate unsteady flow conditions, height-dependent wind profiles, reactive pollutant transport, and terrain influences to enhance its applicability to complex environmental systems.

7. References

[1] Arya, S. P. (1999). Air Pollution Meteorology and Dispersion. Oxford University Press.
[2] Baklanov, A., & Borrego, C. (2006). Urban Air Pollution Modelling and Management. Springer.
[3] El-Khattam, W., & Reitz, R. D. (2012). "Analytical Solutions for Pollutant Dispersion in the Atmospheric Boundary Layer." Atmospheric Environment, 60, 440–449.
[4] Esau, I., & Grachev, A. (2007). "Turbulent Prandtl Number and Atmospheric Dispersion in the Stable Boundary Layer." Boundary-Layer Meteorology, 125(3), 373–385.

[5] Hanna, S. R., Briggs, G. A., & Hosker, R. P. (1982). Handbook on Atmospheric Diffusion. U.S. Department of Energy, Technical Information Center, Oak Ridge.
[6] Holmes, N.S., & Morawska, L. (2006). A review of dispersion modelling and its application to the dispersion of particles. Atmospheric Environment, 40(30), 5902-5928.
[7] Holton, J. R. (2004). An Introduction to Dynamic Meteorology (4th ed.). Elsevier Academic Press.
[8] Jacobson, M. Z. (2002). Atmospheric Pollution: History, Science, and Regulation. Cambridge University Press.
[9] Kumar, P., & Gulia, S. (2017). "Modeling of Urban Air Quality: Current Approaches and Future Challenges." Journal of Environmental Management, 196, 646–663. https://doi.org/10.1016/j.jenvman.2017.03.020
[10] Nappo, C. J. (2002). An Introduction to Atmospheric Gravity Waves. Academic Press.
[11] Pasquill, F., & Smith, F. B. (1983). Atmospheric Diffusion (3rd ed.). Ellis Horwood Ltd.
[12] Rao, S. T. (2003). "Air Quality Models: A Review." Atmospheric Environment, 37(7), 1039–1059.
[13] Raza, M., & Khan, M. (2021). An Analytical Model for Point Source Pollutants in the Atmospheric Environment, Mathematical Theory and Modeling, 11(2), 9–16.
[14] Seinfeld, J. H., & Pandis, S. N. (2016). Atmospheric Chemistry and Physics: From Air Pollution to Climate Change (3rd ed.). Wiley-Interscience.
[15] Turner, D. B. (1994). Workbook of Atmospheric Dispersion Estimates: An Introduction to Dispersion Modeling (2nd ed.). CRC Press.
[16] Venkatram, A., & Wyngaard, J. C. (1988). Lectures on Air Pollution Modeling. American Meteorological Society.
[17] Zannetti, P. (1990). Air Pollution Modeling: Theories, Computational Methods and Available Software. Springer.

8.Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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